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m-Qubit states embedded in Clifford algebras $\mathfrak{Cl}_{2m}$

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Abstract
The quantum theory of a finite quantum system with L degrees of freedom is usually set up by associating it with a Hilbert space $\mathfrak{H}$ of dimension $d(L)$ and identifying observables and states in the matrix algebra $\mathfrak{M}_d(\mathbb{C})$. For the case $d = 2^m$, m integer, this algebra can be identified with the Clifford algebra $\mathfrak{Cl}_{2m}$. The case of $d = 2^m$ dimensions is simply realized by a system with m dichotomic degrees of freedom, an m-qubit system for instance. The physically relevant new point is the appearance of a new (symmetry-?)group $SO(2m)$. A possible interpretation of the space in which this group operates is proposed. It is shown that the eigenvalues of m-qubit-type states only depend on $SO(2m)$-invariants. We use this fact to determine state parameter domains (generalized Bloch spheres) for states classified as $SO(2m)$-tensors. The classification of states and interactions of components of a physical m-qubit system as k-tensors and pseudotensors ($0 \leqslant k \leqslant m$) leads to rules similar to those found in elementary quantum mechanics. The question of electromagnetic interactions is shortly broached. We sketch, pars pro toto, a graphical interpretation of tensor contractions appearing in perturbative expansions.

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1. Introduction

The use of algebraic notions for uncovering structures in state spaces of m-qubit systems has found some attention in the literature [1, 2, 4]. It is of interest to present the results of the former two papers in a simplified form which however yields the essential results. This might be useful to elucidate differences of this approach to the concept pursued in this paper:

m-qubit states ρ are embedded in the 2^{2m}-dimensional $\mathbb{R}$-linear Hilbert space $\mathfrak{H}$ of Hermitian $2^m \times 2^m$ matrices; the roots $\{x_i\}$ of the characteristic polynomial

$$P_{2^m} = \det(\rho - x\mathbb{I}) = \sum_{i=0,\dots,2^m} (-1)^{2^m - i} A_i x^i$$

are related to the coefficients A_i in the well-known manner ($N := 2^m$)

$$\begin{aligned}
A_N &= 1 \\
A_{N-1} &= \sum_{i=1,\dots,N} x_i \\
A_{N-3} &= \sum_{i,j=1,\dots,N}^{\text{all pairs}\,\{i,j\}} x_i x_j \\
A_{N-4} &= \sum_{i,j,k=1,\dots,N}^{\text{all triples}\,\{i,j,k\}} x_i x_j x_k \\
&\vdots \\
A_0 &= x_1 x_2 \cdots x_{N-1} x_N.
\end{aligned}$$

Of course the eigenvalues $\{x_i\}$ of the state ρ and the coefficients $\{A_i\}$ are $SU(2^m)$ invariants. Hence the above relations can be rewritten in terms of powers of $\mathrm{tr}(\rho^k)$, $k = 1, \dots, 2^m$. We have

$$\begin{aligned}
A_N &= 1 \\
A_{N-1} &= \mathrm{tr}(\rho) \\
A_{N-3} &= (\mathrm{tr}(\rho)^2 - \mathrm{tr}(\rho^2))/2 \\
A_{N-4} &= (\mathrm{tr}(\rho)^3 + 2\mathrm{tr}(\rho^3) - 3\mathrm{tr}(\rho^2)\,\mathrm{tr}(\rho))/6 \\
A_{N-5} &= (\mathrm{tr}(\rho)^4 - 6\mathrm{tr}(\rho^4) + 8\mathrm{tr}(\rho)\,\mathrm{tr}(\rho^3) + 3(\mathrm{tr}(\rho^2))^2 - 6\mathrm{tr}(\rho^2)(\mathrm{tr}(\rho))^2)/24 \\
&\vdots \\
&\vdots \\
A_0 &= \mathrm{e}^{\mathrm{tr}(\log(\rho))}.
\end{aligned}$$

These relations serve to express the Descartes necessary and sufficient conditions (ρ is Hermitian)

$$A_i \geqslant 0 \qquad \text{for} \quad i = 0, \dots, N$$

for the positivity of ρ in terms of $2^m - 1$ parameters

$$\begin{aligned}
&\mathrm{tr}(\rho^k), && k = 2, \dots, 2^{2m}; \\
&\mathrm{tr}(\rho) = 1 && \text{(for normalization)}
\end{aligned}$$

instead of the original $2^{2m} - 1$ matrix elements $\rho_{i,j}$. As we see the notion of $SU(2^m)$-invariance leads to conceptual simplifications. For example for state configurations close to a microcanonical distribution terms $\mathrm{tr}(\rho^k)$ may be neglected (k small but sufficiently large) in determining approximate state domains for sufficiently large numbers of degrees of freedom.

On the other hand it should be noted that the $SU(2^m)$-generation of states from their spectrum ($\Lambda := \{\lambda_i, i = 1, \dots, 2^m\} \longrightarrow \rho_{\mathrm{diag}}$)

$$\rho_\Lambda = U^* \rho_{\mathrm{diag}} U, \qquad U \in SU(2^m)$$

seems, physically seen, not to be particularly suited e.g. to distinguish entangled and non-entangled states in the classes ρ_Λ.

We intend to pursue quite a different line of argumentation. As the Hilbert space of an m-qubit system we choose to take

$$(\mathbb{C}^2)^{\otimes m} = \mathbb{C}^{2^m}; \tag{1}$$

sets of observables are in $\mathfrak{M}_{2^m}(\mathbb{C})$, the algebra of complex $2^m \times 2^m$ matrices. The key observation which we are going to exploit is that $\mathfrak{M}_{2^m}(\mathbb{C})$ can be identified with a Clifford algebra $\mathfrak{Cl}(\mathbb{V}, \mathbf{Q})$ generated by a $2m$-dimensional vector space $\mathbb{V}$ and a quadratic form Q on it.

Clifford algebras have a rich structure which lends itself to a new analysis of various aspects of m-qubit states and their observables. The natural basis given in terms of generalized Dirac matrices is useful for the calculations of domains for state parameters [4]. New insight will be gained from an embedding of the spin group $\mathfrak{S}\mathrm{pin}(\mathbb{V}, Q)$ into $\mathfrak{Cl}(\mathbb{V}, Q)$. It acts in $\mathfrak{Cl}(\mathbb{V}, Q)$ and on $\mathbb{V}$ under its homomorphic image $SO(\mathbb{V}, Q)$. The Hilbert space $\mathbb{C}^{2^m}$ of the m-qubit system can be decomposed into two non-equivalent half-spin representations of $\mathfrak{S}_{\mathrm{pin}}(\mathbb{V}, Q)$ of dimensions 2^{m-1}. The sets of pure and mixed states are given as density matrices and identified as Hermitian positive elements of $\mathfrak{Cl}$. The set of all density matrices is a generalization of the Bloch sphere for $m = 1$, a closed and convex subset of $\mathfrak{Cl}$.

Speaking in less formal terms we observe the following features:

- The identification of $\mathfrak{M}_{2^m}(\mathbb{C})$ (in which states are represented as the subset of Hermitian, positive and normalized matrices ($\rightarrow$ density matrices) and observables as linear operators on it) with the corresponding Clifford algebra leads us, and that in a sense is the main point, to the introduction of the real linear space $\mathbb{V} = \mathbb{R}^{2m}$ and the group SO_{2m} acting on it. The filtration and the vector space isomorphisms of Clifford algebras to be described below yield an alternative view of states as (direct sums of) antisymmetric tensors in the exterior algebra $\Lambda^*\mathbb{V}$.
- A possible physical interpretation of the space $\mathbb{V}$ can be given along the following lines. We write m-qubit states in a basis generated by m-fold direct products of qubit states: the up–down projections in each of the m qubits of the m-qubit, $2m$ altogether[1], represent the coordinates of a vector in $\mathbb{V}$. These coordinates can be given an interpretation as classical probabilities, the space $\mathbb{V}$ serves as support for probability distributions the motion of which is determined by classical equations.
- Our construction can be seen as an analog to the well-known discription of relativistic point particles, the Dirac theory. There $\mathbb{V}$ is the Minkowsky spacetime continuum, $SO(3, 1)$, the Lorentz group, stands for SO_{2m} in the m-qubit case, the Dirac algebra for $\mathfrak{Cl}(\mathbb{V}, Q)$.
- The dual view of m-qubit states as elements of the exterior algebra $\Lambda^*\mathbb{V}$ brings us to the important observation that $SU(2^{2m})$ invariant m-qubit observables can be expressed in terms of SO_{2m} invariants which is considering dimensions ($2m$ to 2^{2m}) a substantial simplification for answering many questions. The question of parameter domains for m-qubit states is an example.

The points we shall address in this paper are first to construct explicitly parameter domains for 2- and 3-partite systems for specific tensor configurations thus extending the results obtained in an earlier publication. Secondly a graphical representation of m-qubit states and the interaction between their constituents is introduced.

[1] Actually only $2m - 1$ because of normalization and a projective formulation seems to be in place, to avoid unnessesary precision and hence complication we refrain from spelling out details.

2. Clifford algebras

In this section we collect relevant definitions and relations [3] which we need when discussing properties of *m*-qubit states interpreted as elements of a Clifford algebra. Uncovering the algebraic structure of an *m*-qubit state will lead to simplifications when it comes to explicit realizations of the latter, to novel ansatz in establishing novel, physically justified approximation schemes. Furthermore and most importantly, a well-founded formulation of the dynamics of *m*-qubit systems, equations of motion of a Dirac equation type will emerge. Needless to say, we shall not attempt to present a self-contained and sufficiently complete presentation of Clifford algebras and their geometries but rather restrict ourselves to sketch a coherent and naturally self-contained reminder of the main facts pertinent to our story, *medias in res*.

Let $\mathbb{V}$ be a vector space over a field $\mathbb{K}$ (we assume once and forever $\mathrm{char}(\mathbb{K}) = 0$, i.e. we assume $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and exclude the critical case of $\mathrm{char}(\mathbb{K}) = 2$) and $Q : \mathbb{V} \longrightarrow \mathbb{K}$ a quadratic form on $\mathbb{V}$.

A Clifford algebra $\mathfrak{Cl}(\mathbb{V}, Q)$ is then an associative algebra over $\mathbb{K}$ with unity $\mathbb{I}$ together with a linear map

$$\iota_1 : \mathbb{V} \longrightarrow \mathfrak{Cl}(\mathbb{V}, Q)$$

defined by the universal property that, an associative $\mathbb{K}$-algebra $\mathfrak{A}$ and a linear map

$$\iota_2 : \mathbb{V} \longrightarrow \mathfrak{A} \qquad \text{with}$$
$$\iota_2(v)^2 = Q(v)\mathbb{I} \qquad \text{for all} \quad v \in \mathbb{V}$$

given, there is a unique algebra homomorphism

$$\kappa : \mathfrak{Cl}(\mathbb{V}, Q) \longrightarrow \mathfrak{A}$$

subject to the condition that the diagram

$$\begin{tikzcd} \mathbb{V} \arrow[r, "\iota_1"] \arrow[d, "\iota_2"'] & \mathfrak{Cl}\,(V, Q) \arrow[dl, dashed, "\kappa"] \\ \mathfrak{A} & \end{tikzcd}$$

commute, i.e. $\kappa \circ \iota_1 = \iota_2$.

This characterization of Clifford algebras leads to structural insight. Given a morphism

$$\eta : (\mathbb{V}, Q) \longrightarrow (\mathbb{V}', Q'),$$

i.e. a $\mathbb{K}-$linear map leaving the quadratic form Q invariant, then there is, we conclude, an induced homomorphism (an isomorphism if η is bijective)

$$\tilde{\eta} : \mathfrak{Cl}(\mathbb{V}, Q) \longrightarrow \mathfrak{Cl}(\mathbb{V}', Q').$$

Furthermore, given

$$\eta' : (\mathbb{V}', Q') \longrightarrow (\mathbb{V}'', Q'')$$

we have

$$\widetilde{\eta \circ \eta'} = \tilde{\eta} \circ \tilde{\eta}'.$$

In this way the Q-orthogonal group $O(\mathbb{V}, Q) = \{\eta \in GL(\mathbb{V}) | \eta^* Q = Q\}$ lifts into the group of automorphisms

$$O(\mathbb{V}, Q) \longrightarrow \mathrm{Aut}(\mathfrak{Cl}(\mathbb{V}, Q)). \tag{2}$$

The linear map

$$\epsilon : v \mapsto -v$$

on $\mathbb{V}$ leaves Q invariant and extends to an automorphism

$$\tilde{\epsilon} : \mathfrak{Cl}(\mathbb{V}, Q) \longrightarrow \mathfrak{Cl}(\mathbb{V}, Q).$$

Since $\tilde{\epsilon}^2 = \mathbb{I}$ the Clifford algebra decomposes into even and odd parts

$$\mathfrak{Cl}(\mathbb{V}, Q) = \mathfrak{Cl}(\mathbb{V}, Q)^0 \oplus \mathfrak{Cl}(\mathbb{V}, Q)^1$$
$$\mathfrak{Cl}(\mathbb{V}, Q)^j = \{\gamma \in \mathfrak{Cl}(\mathbb{V}, Q) | \tilde{\epsilon}(\gamma) = (-1)^j \gamma\} \longrightarrow$$

$\mathfrak{Cl}(\mathbb{V}, Q)$ is a $\mathbb{Z}_2$-graded algebra.

A more explicit realization of the latter will be described now. Let

$$\mathfrak{T}(\mathbb{V}) := \sum_{s=0}^{\infty} \bigotimes{}^s \mathbb{V}$$

denote the tensor algebra of $\mathbb{V}$ and impose the defining map ι_2 by factoring the two-sided ideal $\mathfrak{I}_Q$ generated by the form $v \otimes v - Q(v)\mathbb{I}^{(2)}$,

$$\mathfrak{I}_Q = \sum_{i,j} t_i \otimes (v \otimes v - Q(v)\mathbb{I}^{(2)}) \otimes t_j$$
$$t_k \in \bigotimes{}^k \mathbb{V} \subset \mathfrak{T}(\mathbb{V}) \text{have (pure) degree } k, \quad k = i, j$$
$$\mathbb{I}^{(2)} \text{ is the unity for } k = 2$$

to define

$$\mathfrak{Cl}(\mathbb{V}, Q) := \mathfrak{T}(\mathbb{V})/\mathfrak{I}_Q.$$

The canonical projection

$$\pi_Q : \mathfrak{T} \longrightarrow \mathfrak{Cl}(\mathbb{V}, Q)$$

provides us with a natural embedding

$$\iota_1 : \mathbb{V} \hookrightarrow \mathfrak{Cl}(\mathbb{V}, Q).$$

An important role in our argumentations plays the fact that the Clifford algebra is a filtered algebra. This becomes clear when we consider the relationship between the Clifford algebra $\mathfrak{Cl}(\mathbb{V}, Q)$ over $\mathbb{V}$, Q and the exterior algebra $\Lambda^*\mathbb{V}$. The tensor algebra has a natural filtration

$$\bar{\mathfrak{F}}^0 \subset \bar{\mathfrak{F}}^1 \subset \bar{\mathfrak{F}}^2 \subset \cdots \subset \mathfrak{T}(\mathbb{V})$$

with

$$\bar{\mathfrak{F}}^k = \sum_{l \leqslant k} \bigotimes{}^l \mathbb{V}.$$

Taking

$$\mathfrak{F}^k := \pi_Q(\bar{\mathfrak{F}}^k)$$

the filtration

$$\mathfrak{F}^0 \subset \mathfrak{F}^1 \subset \mathfrak{F}^2 \subset \cdots \subset \mathfrak{Cl}(\mathbb{V}, Q)$$

obtains.

We furthermore have

$$\mathfrak{F}^i \otimes \mathfrak{F}^j = \mathfrak{F}^{i+j};$$

a multiplication which descends to the map

$$\mathfrak{F}^i/\mathfrak{F}^{i-1} \otimes \mathfrak{F}^j/\mathfrak{F}^{j-1} \longrightarrow \mathfrak{F}^{i+j}/\mathfrak{F}^{i+j-1}.$$

The algebra spanned by[2] the quotients $\mathfrak{F}^{*k} := \mathfrak{F}^k/\mathfrak{F}^{k-1}$

$$\mathfrak{F}^* := \bigoplus_{k=0} \mathfrak{F}^{*k} \tag{3}$$

is called the associated graded algebra. The

- Proposition
  For any quadratic form Q there is a canonical vector isomorphism ι

  $$\iota : \Lambda^*\mathbb{V} \longrightarrow \mathfrak{Cl}(\mathbb{V}, Q)$$

  compatible with the filtration described above. The associated graded algebra $\mathfrak{F}^*$ is naturally isomorphic to the exterior algebra $\Lambda^*\mathbb{V}$

was used in [4] for the classification of quantum states according to their tensor character.

A $\mathbb{Z}_2$-graded tensor product for Clifford algebras is introduced as

$$(\mathfrak{F} \hat{\otimes} \mathfrak{G})^0 = \mathfrak{F}^0 \otimes \mathfrak{G}^0 + \mathfrak{F}^1 \otimes \mathfrak{G}^1$$
$$(\mathfrak{F} \hat{\otimes} \mathfrak{G})^1 = \mathfrak{F}^0 \otimes \mathfrak{G}^1 + \mathfrak{F}^1 \otimes \mathfrak{G}^0.$$

For the formulation of a cluster decomposition of m-qubit states we need the following observation:

- Let

  $$\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$$

  denote a decomposition in Q-orthogonal subspaces $\mathbb{V}_{1,2}$.
  Then the Clifford algebra $\mathfrak{Cl}(\mathbb{V}, Q)$ is isomorphic to the graded product
  $\mathfrak{Cl}(\mathbb{V}_1, Q_1) \hat{\otimes} \mathfrak{Cl}(\mathbb{V}_2, Q_2)$ of Clifford algebras over $\mathbb{V}_{1,2}$ (the $Q_{1,2}$ are of course the restrictions of Q to $\mathbb{V}_{1,2}$)

  $$\iota : \mathfrak{Cl}(\mathbb{V}, Q) \longrightarrow \mathfrak{Cl}(\mathbb{V}_1, Q_1) \hat{\otimes} \mathfrak{Cl}(\mathbb{V}_2, Q_2).$$

The tensor algebra $\mathfrak{T}(\mathbb{V})$ has an involution defined by reversal of the order of elements in products

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \longrightarrow v_k \otimes \cdots \otimes v_2 \otimes v_1.$$

This involution preserves the ideal $\mathfrak{I}_Q$ and hence descends to a map, the transposition

$$(\,)^t : \mathfrak{Cl}(\mathbb{V}, Q) \longrightarrow \mathfrak{Cl}(\mathbb{V}, Q). \tag{4}$$

This map is an anti-automorphism

$$(c_1 \cdot c_2)^t = c_2^t c_1^t.$$

A bilinear form (scalar product) is defined from Q in the usual manner

$$\langle w, v\rangle := (Q(w+v) - Q(w) - Q(v))/2.$$

The defining relation $\iota_2(v)^2 = Q(v)\mathbb{I}$ then takes its more familiar form of an anticommutation relation

$$\iota_2(v) \cdot \iota_2(w) + \iota_2(w) \cdot \iota_2(v) = 2\langle v, w\rangle\mathbb{I}.$$

[2] Spanned by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}, v_{i_l} \in \mathbb{V}$.

Assuming from now on whenever it does not seem to lead to confusion the identification of notation of image and preimage, we write the simplified version of this equation as

$$vw + wv = 2\langle v, w\rangle \mathbb{I}. \tag{5}$$

Extending Q to tensor products of $v_i \in \mathbb{V}$ [3]

$$Q(v_1 \otimes v_2 \otimes \cdots \otimes v_k) := Q(v_1)Q(v_2)\cdots Q(v_k) \qquad v_j \in \mathbb{V}$$

which is simply the scalar part of $(v_1 \otimes v_2 \otimes \cdots \otimes v_k)^t \otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_k)$. This leads to the extended definition ($\langle s\rangle$ denotes the scalar part of $s \in \mathfrak{Cl}(\mathbb{V}, Q)$)

$$Q(r) := \langle r^t r\rangle \qquad r \in \mathfrak{Cl}(\mathbb{V}, Q).$$

The associated bilinear form then is written as

$$\langle r, s\rangle = \langle r^t s\rangle.$$

Subgroups of the Clifford algebra are of key importance for the formulation of the dynamics of m-qubit systems. The subset

$$\mathfrak{Cl}(\mathbb{V}, Q)^{\{\times\}} := \{r \in \mathfrak{Cl}(\mathbb{V}, Q) \mid \exists r^{-1}, rr^{-1} = r^{-1}r = 1\}$$

constitutes obviously a group, *the multiplicative group of units in the Clifford algebra*, its dimension is

$$\dim(\mathfrak{Cl}(\mathbb{V}, Q)^{\{\times\}}) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The Clifford algebra is the Lie algebra of this group

$$\mathfrak{cl}^{\{\times\}}(\mathbb{V}, Q) = \mathfrak{Cl}(\mathbb{V}, Q),$$

its bracket is $[r, s] = rs - sr$.

This group acts naturally as an automorphism of $\mathfrak{Cl}(\mathbb{V}, Q)$ and we have the homomorphism, the adjoint representation

$$\mathrm{Ad} : \mathfrak{Cl}(\mathbb{V}, Q)^{\{\times\}} \longrightarrow \mathrm{Aut}(\mathfrak{Cl}(\mathbb{V}, Q))$$

which is given as

$$\mathrm{Ad}_r(c) = r^{-1}c\tilde{\epsilon}(r)$$

$$r \in \mathfrak{Cl}(\mathbb{V}, Q)^{\{\times\}}, c \in \mathfrak{Cl}(\mathbb{V}, Q)$$

$\tilde{\epsilon}$ is the reflection introduced above :$\tilde{\epsilon}(r) = (-1)^{\mathrm{degree(r)}}r$.

Take $v, w \in \mathbb{V} \subset \mathfrak{Cl}(\mathbb{V}, Q)$, $Q(v) \neq 0$ and calculate

$$\begin{aligned}\mathrm{Ad}_v(w) &= -vwv^{-1} = -vwv/Q(v) = (v^2w - 2\langle v, w\rangle v)/Q(v)\\ &= w - 2\frac{\langle v, w\rangle}{Q(v)}v.\end{aligned}$$

Geometrically speaking, $\tilde{\epsilon}(\mathrm{Ad}_v(\cdot))$ is the reflection of $(\cdot)$ across the hyperplane $\langle \cdot, v\rangle = 0$.

Let us now consider the subgroup of elements $r \in \mathfrak{Cl}(\mathbb{V}, Q)^{\{\times\}}$ with

$$\mathrm{Ad}_r(\mathbb{V}) = \mathbb{V},$$

the Clifford group $\mathfrak{G}_{Cl}(\mathbb{V}, Q)$, and observe from the above equation that $\mathbb{V} \subset \mathfrak{G}_{Cl}(\mathbb{V}, Q)$ and that for $v, w \in \mathbb{V}$, $Q(v) \neq 0$ we have

$$(\mathrm{Ad}_v^* Q)(w) := Q(\mathrm{Ad}_v(w)) = Q(w),$$

i.e. the transformation Ad_v leaves the quadratic form Q invariant.

[3] We tacitly introduce here a basis in $\mathbb{V}$, $\{e_1, e_2, \ldots, e_n\}, n < \infty$ is its dimension, $\langle e_i, e_j\rangle = \delta_{ij}$. The Clifford identity requires $e_i \otimes e_j + e_j \otimes e_i = 0$ for $i \neq j$. A basis for $\mathfrak{Cl}(\mathbb{V}, Q)$ is then given as $\{e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_k} | 1 \leqslant i_1 \leqslant i_2 \cdots \leqslant i_k \leqslant n, 0 \leqslant k \leqslant n\}$.

$\mathfrak{G}_{Cl}(\mathbb{V}, Q)$ maps onto the Q-orthogonal group $O(\mathbb{V}, Q)$ and the kernel of this map is not larger than $\mathbb{K}^*$, the group of non-zero multiples of $\mathbb{I}$

$$\mathfrak{G}_{Cl}(\mathbb{V}, Q) \xrightarrow{\mathrm{Ad}} O(\mathbb{V}, Q) \qquad \text{with} \quad O(\mathbb{V}, Q) = \{t \in GL(\mathbb{V}) \mid t^*Q = Q\}.$$

Splitting the Clifford group into even and odd parts, $\mathfrak{G}_{Cl}(\mathbb{V}, Q)^0$ (is a subgroup), and $\mathfrak{G}_{Cl}(\mathbb{V}, Q)^1$ we subsume these findings into two exact sequences

$$\mathbb{I} \longrightarrow \mathbb{K}^* \longrightarrow \mathfrak{G}_{Cl}(\mathbb{V}, Q) \xrightarrow{\mathrm{Ad}} O(\mathbb{V}, Q) \longrightarrow \mathbb{I}$$
$$\mathbb{I} \longrightarrow \mathbb{K}^* \longrightarrow \mathfrak{G}_{Cl}(\mathbb{V}, Q)^0 \xrightarrow{\mathrm{Ad}} SO(\mathbb{V}, Q) \longrightarrow \mathbb{I}.$$

A spinor norm N can be defined as

$$N(r) := r^t r,$$

and yields, restricted to the Clifford group, $\mathfrak{G}_{Cl}(\mathbb{V}, Q)$, a homomorphism

$$N : \mathfrak{G}_{Cl} \longrightarrow \mathbb{K}^*.$$

We now define the groups $\mathfrak{Pin}$ and $\mathfrak{Spin}$ as the groups generated not by $\mathbb{V}$ but by the generalized unit spheres

$$\mathfrak{Pin}(\mathbb{V}, Q) = \{r \in \mathfrak{G}_{Cl} | N(r) = \pm\mathbb{I}\},$$

the group $\mathfrak{Spin}$ is the even part of $\mathfrak{Pin}$

$$\mathfrak{Spin}(\mathbb{V}, Q) = \mathfrak{Pin}(\mathbb{V}, Q) \cap \mathfrak{Cl}(\mathbb{V}, Q)^0.$$

There is a homomorphism from $\mathfrak{Pin}$ to $O(\mathbb{V}, Q)$, its kernel is $\{+\mathbb{I}, -\mathbb{I}\}$. Again we summarize the last paragraphs in the following exact sequences:

$$\mathbb{I} \longrightarrow \mathfrak{Z} \longrightarrow \mathfrak{Pin}(\mathbb{V}, Q) \xrightarrow{\mathrm{Ad}} O(\mathbb{V}, Q) \longrightarrow \mathbb{I}$$
$$\mathbb{I} \longrightarrow \mathfrak{Z} \longrightarrow \mathfrak{Spin}(\mathbb{V}, Q) \xrightarrow{\mathrm{Ad}} SO(\mathbb{V}, Q) \longrightarrow \mathbb{I}$$

where

$$\mathfrak{Z} = \begin{cases} \{\pm\sqrt{1}\} \quad \mathbb{I} \\ \{\pm\sqrt{1}, \pm\sqrt{-1}\}\mathbb{I} \end{cases} \qquad \text{for} \quad \mathbb{K} = \mathbb{R} \quad \text{or} \quad \mathbb{K} = \mathbb{C} \quad \text{respectively.}$$

The group $\mathfrak{Spin}(\mathbb{V}, Q)$ is a cover of the special orthogonal group $SO(\mathbb{V}, Q)$

$$\mathfrak{Spin}(\mathbb{V}, Q) \longrightarrow \mathfrak{Spin}(\mathbb{V}, Q)/\{+\mathbb{I}, -\mathbb{I}\} \xrightarrow{\approx} SO(\mathbb{V}, Q)$$

where the first arrow indicates the covering homomorphism.

3. Tensor states

In this section we turn to a classification of m-qubit states which has its origin in the identification of the underlying linear space, the space $\mathfrak{M}_{2^m}(\mathbb{C})$ of $2^m \times 2^m$ complex matrices, with the complex Clifford algebra $\mathfrak{Cl}_{2m}(\mathbb{V}, Q)$ discussed above. The filtration (3) then leads to a classification of m-qubit states as $SO(2m)$-tensors (and pseudotensors) of degree l, $0 \leqslant l \leqslant m$.

Now take the Euclidean quadratic form

$$Q(v) = v^2$$

and construct a basis of $\mathfrak{Cl}_{2m}(\mathbb{V}, Q)$. The anticommutation relation (5) has 2^{2m}-dimensional representations which we denote by

$$\left\{\Gamma_i^{\{2m,1\}} \,\middle|\, i = 1, \ldots, 2m\right\} \tag{6}$$

where the $\Gamma_i^{\{2m,1\}}$ are traceless, Hermitian matrices with

$$\Gamma_i^{\{2m,1\}} \cdot \Gamma_j^{\{2m,1\}} + \Gamma_j^{\{2m,1\}} \cdot \Gamma_i^{\{2m,1\}} = 2\delta_{i,j} \qquad i, j = 1, \ldots, 2m. \tag{7}$$

From these matrices we construct

$$\begin{aligned}&\text{a basis in the } 2^{2m}\text{-dimensional the representation space} = \\ &\bar{\mathfrak{B}} = \{\Gamma^{\{2m,0\}}, \Gamma^{\{2m,1\}}, \ldots, \Gamma^{\{2m,2m\}}\}\end{aligned} \tag{8}$$

with

$$\Gamma^{\{2m,k\}} = \left\{(-i)^{S_g(k)}\Gamma_{i_1}^{\{2m,1\}} \cdot \Gamma_{i_2}^{\{2m,1\}} \cdots \Gamma_{i_k}^{2m,1} \middle| i_1 < i_2, \cdots < i_k, 1 \leqslant i_l \leqslant 2m\right\}$$

with

$$S_g(k) = \begin{cases} 0 & \text{for} \quad k \bmod(4) = 0, 1 \\ 1 & \text{for} \quad k \bmod(4) = 2, 3 \end{cases} \quad \text{and} \quad 1 \leqslant k \leqslant 2m$$

$$\Gamma^{\{2m,0\}} = \mathbb{I}, \qquad \text{the unit in the representation space.}$$

Introducing the notion of pseudotensors by first constructing the 'pseudo-scalar unit'

$$\Gamma\{5\}^{\{2m\}} = (-i)^{S_g(2m)}\Gamma_1^{\{2m,1\}} \cdot \Gamma_2^{\{2m,1\}} \cdots \Gamma_{2m}^{\{2m,1\}} \tag{9}$$

which anti-commutes with the 'generalized Dirac' matrices

$$\Gamma^{\{2m,1\}} \quad \Gamma_j^{\{2m,1\}} \cdot \Gamma\{5\}^{\{2m\}} + \Gamma_j^{\{2m,1\}} \cdot \Gamma\{5\}^{\{2m\}} = 0, \qquad j = 1, \ldots, 2m, \tag{10}$$

we simplify matters by defining

$$\begin{aligned}
\tilde{\Gamma}^{\{2m,m-1\}} &:= -i\Gamma\{5\}^{\{2m\}} \cdot \Gamma^{\{2m,m-1\}} \\
\tilde{\Gamma}^{\{2m,m-2\}} &:= -i\Gamma\{5\}^{\{2m\}} \cdot \Gamma^{\{2m,m-2\}} \\
&\vdots
\end{aligned} \tag{11}$$

$$\tilde{\Gamma}^{\{2m,0\}} := -i\Gamma\{5\}^{\{2m\}} \cdot \Gamma^{\{2m,0\}} \tag{12}$$

and write the basis of our calculations as

$$\begin{aligned}\mathfrak{B} = \{&\Gamma^{\{2m,0\}}, \Gamma^{\{2m,1\}}, \ldots, \Gamma^{\{2m,m-1\}}, \Gamma^{\{2m,m\}}, \\ &\tilde{\Gamma}^{\{2m,m-1\}}, \ldots, \tilde{\Gamma}^{\{2m,2m-1\}}, \tilde{\Gamma}^{\{2m,0\}}\}\end{aligned} \tag{13}$$

$$:= \{\gamma^{\{2m,i\}} | i = 0, \cdots, 2m\}. \tag{14}$$

Note that $\gamma^{\{2m,m+k\}}$ is a (m–k)-tensor, $k = 1, \ldots, m$. Tensors and pseudotensors differ in their behaviour under $(\cdot)^t$. States written in this basis then read

$$\rho := \frac{1}{2^{2m}} \sum_{i=0}^{2m} \alpha^{\{i\}} \cdot \gamma^{\{2m,i\}}. \tag{15}$$

Tensor states in particular are given as

$$\rho^{\{i\}} = \frac{1}{2^{2m}} (\gamma^{\{2m,0\}} + \alpha^{\{i\}} \cdot \gamma^{\{2m,i\}}), \tag{16}$$

the $\alpha^{\{i\}} \in \Lambda^*\mathbb{R}^{2m}$ are antisymmetric tensors, normalization requires

$$\text{trace}(\rho) = 1 \quad \Longleftrightarrow \quad \alpha^{\{0\}} = 1.$$

The dot indicates tensor contraction

$$\alpha^{\{i\}} \cdot \gamma^{\{2m,i\}} = \sum_{k_1,\ldots,k_i} \alpha^{\{i\}}_{k_1,k_2,\ldots,k_i} \gamma^{\{2m,i\}}_{k_1,k_2,\ldots,k_i}.$$

We determine the domains for the (real) tensor elements $\alpha^{\{i\}}_{k_1,\ldots,k_i}$ guaranteeing positivity and normalization of a state ρ. The pivot of our calculations is the characteristic polynomial of ρ. The domains can then be deduced from Descartes' rule:

- The characteristic polynomial is written as
$$\begin{aligned}\text{Pol}^{\{2m\}} &= \text{Determinant}(\rho - \lambda\mathbb{I}^{\{2m\}}) \\ &= \sum_{k=0}^{2m} A_k(-\lambda)^k.\end{aligned}$$
- Normalization obviously implies
$$A_1 = 1.$$
- Necessary and sufficient conditions for the positivity of the eigenvalues $\{\lambda_j, j = 1, \ldots, 2m\}$ are
  - (a) The coefficients A_k are such that the $\{\lambda_i\}$ are real, a condition which is guaranteed here by hermiticity of ρ.
  - (b) The coefficients A_k fulfil the inequalities
$$A_k \geqslant 0.$$

We follow the method of calculating the eigenvalues, the probability spectrum, express the latter in terms of $SO(2m)$-invariants and determine the domains of the latter guaranteeing positivity, i.e. the generalized Bloch spheres. The spectra for $m = 2$ and $m = 3$, vectors, 2-tensors and 3-tensors are explicitly calculated. Since the spectra for vectors and pseudovectors, tensors and the corresponding pseudotensors are identical the bases for $m = 2, 3$ Clifford algebras are treated. We have

- $m = 2$
  - Vector:
$$\alpha^{\{1\}} = [\alpha_1, \ldots, \alpha_4]$$
$$R_v := \|v\|^2 = \sum_{i=1}^{4} {\alpha_i}^2.$$
For the spectrum we get
$$\lambda_{1,2} = \tfrac{1}{4}(1 \pm \sqrt{R}) \qquad \lambda_{3,4} = \tfrac{1}{4}(1 \pm \sqrt{R})$$
and observe two-fold degeneracy.
  - 2-tensor
$$\alpha^{\{2\}} = \begin{bmatrix} 0 & \alpha_{1,2} & \ldots & \alpha_{1,2^m} \\ -\alpha_{1,2} & 0 & \ldots & \alpha_{2,2^m} \\ \vdots & & & \\ -\alpha_{1,2^m} & \ldots & \ldots & 0 \end{bmatrix} \quad \text{antisymmetric.}$$
For the spectrum we get

$\{\lambda\} = \left\{\frac{1}{4}(1 \pm \sqrt{R_2 \pm \sqrt{X}})\right\}$
where
$R_2 = \sum_{i<k} \alpha_{i,k}^2$is the $\frac{1}{2}$(Frobenius norm)^2 of $\alpha^{\{2\}}$:
$\|\alpha^{\{2\}}\|_{\mathrm{Frob}}{}^2 = \mathrm{trace}(\alpha^{\{2\}^t} \cdot \alpha^{\{2\}})$,
$(\,)^t$ is the transposition introduced in (4)
$X = 2\big(R_2^2 - \mathrm{trace}\big(\{\alpha^{\{2\}^t} \cdot \alpha^{\{2\}}\}^2\big)/2\big)$,
the spectrum is expressed in terms of traces of tensors in $\mathbb{R}^{2m}$ (see the corresponding formulae in the introduction where traces pertain to $\mathbb{R}^{2^m}$).

- $m = 3$
  - Vector

$\beta^{\{1\}} = \left[\beta_1^{\{1\}}, \beta_2^{\{1\}}, \ldots, \beta_6^{\{1\}}\right]$
$R_v = \sum_{i=1}^{6} \beta_i^{\{1\}2}$
For the spectrum we get
$\lambda_{1,2} = \frac{1}{4}(1 \pm \sqrt{R_v})\ \lambda_{3,4} = \frac{1}{4}(1 \pm \sqrt{R_v})\ \lambda_{5,6} = \frac{1}{4}(1 \pm \sqrt{R_v})$
and observe three − fold degeneracy

  - 2-tensor

$$\beta^{\{2\}} = \begin{bmatrix} 0 & \beta_{1,2} & \ldots & \beta_{1,2^m} \\ -\beta_{1,2} & 0 & \ldots & \beta_{2,2^m} \\ \vdots & & & \\ -\beta_{1,2^m} & \ldots & \ldots & 0 \end{bmatrix} \qquad \text{antisymmetric.}$$

For the spectrum we get
$\{\lambda_{1,\ldots,4}\} = \left\{\frac{1}{8}(1 \pm \sqrt{R_2 \pm \sqrt{X}})\right\}$
$\{\lambda_{5,\ldots,8}\} = \left\{\frac{1}{8}(1 \pm \sqrt{R_2 \pm \sqrt{X}})\right\}$
where
$R_2 = \sum_{i<k} \beta_{i,k}^2$ is the $\frac{1}{2}$(Frobenius norm)^2 of $\beta^{\{2\}}$,
$X = 2\big(R_2^2 - \mathrm{trace}(\beta^{\{2\}^4})/2\big)$,
the spectrum is two-fold degenerate and expressed in terms of traces of tensors in $\mathbb{R}^{2m}$, the same formulae hold as in the $m = 2$ case.

  - 3-tensor

$\delta^{\{3\}} = (\delta_{i,j,k})$ totally antisymmetric
For the spectrum we get
$\{\lambda_{1,\ldots,4}\} = \left\{\frac{1}{8}(1 \pm \sqrt{R_3 \pm \sqrt{X}})\right\}$
$\{\lambda_{5,\ldots,8}\} = \left\{\frac{1}{8}(1 \pm \sqrt{R_3 \pm \sqrt{X}})\right\}$

where

$$R_3 = \sum_{i<j<k} \delta_{i,j,k}^2 =: \frac{1}{3!}\,\mathrm{trace}(\delta^t\cdot\delta)$$

$$X = 2\left(R_3^2 - \frac{1}{3!}\,\mathrm{trace}_{\mathrm{irred}}((\delta^t\cdot\delta)^2)\right).$$

We give an explicit expression for $\mathrm{trace}_{\mathrm{irred}}((\delta^t\cdot\delta)^2)$

$$\mathrm{trace}_{\mathrm{irred}}((\delta^t\cdot\delta)^2) := \sum_{\substack{[k_1,k_2,k_5],[k_2,k_3,k_6],[k_3,k_4,k_5],[k_1,k_4,k_6]\\ \text{antisymmetric}}} \delta_{k_1,k_2,k_5}\delta_{k_2,k_3,k_6}\delta_{k_3,k_4,k_5}\delta_{k_1,k_4,k_6} \tag{17}$$

and define it after the introduction of a graphical interpretation of trace- contractions in the following section.

Ending this section we propose a universal formula for 2-tensors in m-qubit states. Considering the antisymmetry it is convenient to introduce a definition of traces which accounts for it. We define

$$\mathrm{trace}^{\{S\}}() := \frac{1}{l!}\mathrm{trace}()$$

where () is some expression involving (anti)symmetric l-tensors (18)

α_l, $1 \leqslant l \leqslant 2m$, extrapolate from the cases $m = 2, 3$

and propose for all m

- $2^{2(m-2)}$ quadruplets of eigenvalues
- $\lambda = \frac{1}{2^m}\left(1 \pm \sqrt{T_2 \pm \sqrt{\{T_2\}^2 - T_4}}\right)$ (19)

where $T_2 = \mathrm{trace}^{\{S\}}(\alpha^t\cdot\alpha)$ $\qquad T_4 = \mathrm{trace}^{\{S\}}(\{\alpha^t\cdot\alpha\}^2)$.

The same formula holds for 3-tensors ($l = 3$) if we replace T_4 by $T_{4\mathrm{irred}}$.

Finally from (19) we read off the parameter domains for (16) to be a state

$$1 \geqslant T_2 \pm \sqrt{T_2^2 - T_4} \geqslant 0. \tag{20}$$

The inequality

$$T_2^2 \geqslant T_4$$

is just the Schwarz inequality.

4. Presentation in terms of graphs

In the following we shall sketch a possible application of the idea of classifying quantum states as antisymmetric real tensors on a $2m$-dimensional vector space by representing these states as graphs which in term can be used to visualize, e.g. perturbative expansions and thus to relate specific features of interactions to interactive modifications of states and expectation values.

By imbedding states of m-qubit systems in the Clifford algebra $\mathfrak{Cl}_{2m}$ we obtained its filtration and thus a dual representation of states in terms of real, antisymmtric tensors, i.e. elements of the corresponding exterior algebra $\Lambda^*\mathbb{R}^{2m}$.

Its elements are represented as vertices, indices as incoming (upper index) and outgoing (lower index). The distinction of upper and lower indices is introduced here to facilitate book keeping and of course has no geometrical meaning (in specifying the Clifford algebra we

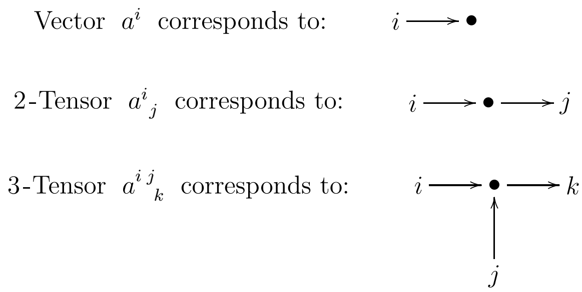


Figure 1. Graphic representation of tensors.

introduced an Euclidean quadratic form $Q(v) = v^2$). The graphical correspondence we draw as follows (figure 1):

Graphs are now built by simply identifying in- and outgoing arrows to obtain the desired index structure. Relevant to the present context is the graphical representation of traces of powers of tensors. Particularly powerful is the graphical representation when a systematic study of interactions, mutual and with external sources, is required for a significant physical description of the underlying phenomena. We begin with a systematic study of traces and then turn to the case when interactions are important. More precisely we draw the graphs for traces of second- and fourth-order powers of 1-, 2-, 3-tensors. In the following we use the Einstein summation convention (figure 2):
repeated upper–lower indices are summed over.

The fact that for the case of 3-vertices only irreducible graphs contribute in equation (19), found in actual calculations for $m = 2, 3$, reminds us of a theorem valid in the quantum field theory of statistical mechanics: a perturbative expansion of the free energy in terms of interactions contains only irreducible graphs. We refrain from pursueing this point any further.

A second point to note is the SO_{2m}-invariance of the graphs introduced above.

Let $\mathfrak{O}^i_j$ denote a SO_{2m}-transformation. Since

$$\mathfrak{O}^i{}_j a^{j\ldots}_{\ldots} = a'^{i\ldots}_{\ldots}$$

upper indices transform with $\mathfrak{O}$,

$$\mathfrak{O}^i{}_j a^{\ldots}_{i\ldots} = a'^{\ldots}_{j\ldots}$$

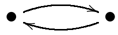
lower indices transform with $\mathfrak{O}^t$

and $\mathfrak{O}^{t\,i}_j \mathfrak{O}^j{}_k = \delta^i{}_k$ by the very definition, the invariance is obvious.

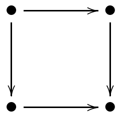
5. Concluding remarks

The identification of the algebra of observables operating in the Hilbert space $\mathfrak{H} = \mathfrak{C}^{2^m}$ associated with an m-qubit system with the Clifford algebra $\mathfrak{Cl}_{2m}$ has various physically interesting consequences for the structure of m-qubit states. In section 1 we collected various concepts and formulae of the theory of Clifford algebras. Of particular interest is the construction of filtrations which as we show in detail leads to a dual representation of states in terms of $SO(2m)$-tensors. As an application we point out that the eigenvalues of a state ρ depend only on $SO(2m)$-invariants, parameter domains expressed in terms of tensor contractions in the dual representation follow. For explicit calculations we realize this dual representation by working in the R-linear subspace of Hermitian matrices in $M_{2^m}(\mathfrak{C})$ whose basis is explicitly constructed from generalized Euclidean Dirac matrices. Explicit equations

– invariant:
$T_2 = a_i^j\, a_j^i$

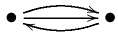


– invariant:
$T_4 = a_j^i\, a_k^j\, a_l^k\, a_i^l$



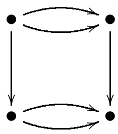
- 3-Tensor: 3-vertex

  – invariant:
  $T_2 = a_{ji}^k\, a_k^{ij}$



  – Two invariants:

  (a) $T_4^{\{red\}} = a^{k_1k_2k_3}\, a^{k_4}{}_{k_3k_2}\, a_{k_1}{}^{k_5k_6}\, a_{k_6k_5k_4}$



  (b) $T_4^{\{irred\}} = a_{k_1k_2k_3}\, a_{k_6}{}^{k_2k_5}\, a^{k_3k_4k_6}\, a^{k_1}{}_{k_4k_5}$

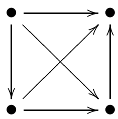


Figure 2. Graphical representation of traces.

for parameter domains, the generalized Bloch spheres, for the cases of 1-, 2-, 3-tensors are given (equation (20)). Geometrically speaking, these domains appear as superpositions of spheres whose radii are determined by the Frobenius norm of the involved tensors.

As a further application we sketch a graphical representation of the tensor contractions occurring in the perturbative expansions of expectation values of observables classified as $SO(2m)$-tensors. The importance of a graphical representation becomes clear when we introduce interactions and develop a graphical picture of the perturbation expansion in terms of interaction Hamiltonians.

The equations of motion, the Schrödinger equation

$$i\dot{\rho} = [H, \rho]$$

or in the context of m-qubits eventually more appropriate, the Lindblad equation

$$\dot{\rho} = -i[H, \rho] + V\rho V^* - \tfrac{1}{2}[\rho, V^*V]_+$$

have the following solution

$$\rho = \mathrm{e}^{\mathrm{i}Ht}\rho(0)\,\mathrm{e}^{-\mathrm{i}Ht},$$

respectively following the modified equation

$$\dot{\rho}_S = V_S\rho_S V_S^* - \tfrac{1}{2}[\rho_S, V_S^* V_S]_+$$

where

$$\rho_S = \mathrm{e}^{\mathrm{i}Ht}\rho\,\mathrm{e}^{-\mathrm{i}Ht} \qquad \text{and} \qquad V_S = \mathrm{e}^{-\mathrm{i}Ht}V\,\mathrm{e}^{\mathrm{i}Ht},$$

the Lindblad equation in the Schrödinger picture.

As usual we split the Hamiltonian into a kinetic and an interactive part

$$H = H_{\mathrm{kin}} + H_{\mathrm{int}} \tag{21}$$

and compute a perturbative expansion (in order not to overload formulae and graphs we restrict ourselves to the Schrödinger case). H_{kin} may contain interaction terms 'dressing' the constituents whose free motion it describes (if the theory allows for it); H_{int} stands for the interaction of these constituents.

$$\rho(t) = \rho_0 + i[H_{\mathrm{int}}, \rho_0] + H_{\mathrm{int}}\rho_0 H_{\mathrm{int}} - \tfrac{1}{2}\left[H_{\mathrm{int}}^2, \rho_0\right] + \cdots \tag{22}$$

where ρ_0 moves with H_{kin}

$$\rho_0(t) = \mathrm{e}^{\mathrm{i}H_{\mathrm{kin}}t}\rho_0(0)\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{kin}}t}.$$

To take advantage of tensor expansions we write H (and any other observable) as

$$H = \sum_{i=0}^{2m} h^{\{i\}}\,\gamma^{\{2m,i\}}. \tag{23}$$

It is immediately seen that perturbative terms lead to a mixing with higher order tensors unless the Hamiltonian is a SO_{2m}-scalar; e.g. a vector state receives by first-order perturbation a $(l+1)$-tensor contribution from an l-tensor term in the Hamiltonian. Tensor components ($[\ldots]_{\mathrm{Anti}}$ denotes anti-symmetrization, note that this is automatic in a Clifford algebra)

$$\alpha'_{[i_1,\ldots,i_k,j_1,\ldots,j_l]_{\mathrm{Anti}}} = \alpha^{\{k\}}_{[i_1,\ldots,i_k]_{\mathrm{Anti}}} h^{\{l\}}_{[j_1,\ldots,j_l]_{\mathrm{Anti}}}$$

appear in the SO_{2m}-expansion of states.

It is well known that the splitting of $H = H_{\mathrm{kin}} + H_{\mathrm{int}}$ is to a large extent arbitrary—part of the interaction (e.g. with external fields) can be included in H_{kin}. We simply assume a division into a scalar, 2-tensor kinetic part and a 4-point interaction

$$H = h_0\mathbb{I} + h^{\{2\}}\cdot\gamma^{\{2m,2\}} + h^{\{4\}}\cdot\gamma^{\{2m,4\}}.$$

For an illustration of a graphical expansion of states we should like to turn to a case of real interest: the electromagnetic interaction. Within the scope of concluding remarks we sketch a rather cursory approach to this question, still emphasizing the main issues. The first step is to attribute a Clifford tensor character to the electromagnetic field. As usual in gauge theory we couple the electromagnetic potential A^i, $i = 1, \ldots, 4$. It is tempting to take the vector for $m = 2$ and write

$$\mathfrak{A} := A^i\gamma_i^{\{4,1\}}.$$

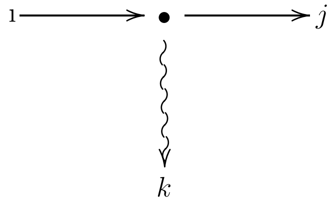


Figure 3. In- and outgoing 'particles' interact with external field.

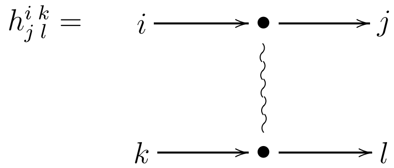


Figure 4. One-photon exchange potential describing the induced 'particle–particle' interaction.

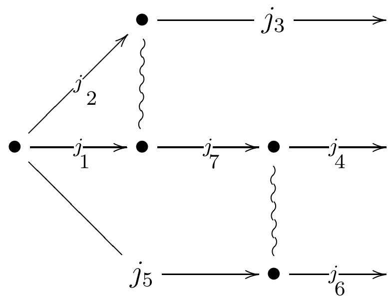


Figure 5. A second-order perturbation of a 3-tensor state.

This object has to interact with a 'matter'-current. We propose

$$H_{\text{matter-em field}} = h^{ijk}\gamma_{ij}^{\{4,2\}}\gamma_k^{\{4,1\}} \tag{24}$$

$$h^{ijk} = \epsilon^{ijki_1j_1k_1}\hat{\alpha}^{\{2\}}_{i_1j_1}A_{k_1}, \tag{25}$$

$\hat{\alpha}^{\{2\}}_{i_1j_1}$ is the current—we are not in the position, the dynamics being not sufficiently far developed here, to discuss immediately arising questions most importantly gauge invariance and current conservation in the context of *m*-qubit interactions.

The antisymmetric 3-tensor h^i_{jk} stands for the Yukawa-type interaction in *m*-qubits and is represented by the following graph (figure 3):

Allowing for in- and outgoing photons, i.e. absorption and emission of photons, and thus simulating a picture with a quantized photon field we now construct, tracing out photons, a 4-point interaction. A graphical representation (figure 4) may suffice.

This interaction Hamiltonian is constructed to describe interactions in 2-qubit systems. To describe electromagnetic interactions in *m*-qubits with $m > 2$ we have to devise a Hamiltonian for pairwise interactions. Let $m = 2m^{\{4\}}$ be even, define partitions of $[1, \ldots, 2m]$ into $m^{\{4\}}$ 4-plets, and construct $m^{\{4\}}$-fold direct products in the following way:

$$H_{\text{int}} = \sum_{\text{all partitions}} \underbrace{\cdots \otimes \mathbb{I}_4 \otimes H^{\{4\}} \otimes \mathbb{I}_4 \cdots}_{\mathrm{m}^{\{4\}}-\text{fold product}}. \tag{26}$$

Such a H_{int} induces interactions between all constituents of an m-qubit; to give an illustration we take the case of a 3-tensor state (figure 5).

Needless to say, further theoretical specifications are necessary to provide a suitable machinery for realistic calculations. We hope to present some further results in a forthcoming publication. Central in our research is the idea described in this paper: embedding states of m-qubits in the Clifford algebra $\mathfrak{Cl}_{2m}$ we were led to using its filtration and the vector space isomorphy with the external algebra $\Lambda^*\mathbb{V}_{2m}$ to get an embedding of states in real, antisymmetric l-tensor spaces $l = 0, \ldots, m$ (we distinguish tensor- and pseudotensor-spaces). A classification of states and their visualization as vertices in a graphical language is obtained. Expanding Hamiltonians in the same way in a Clifford basis, state transitions are characterized by a change of the degree of tensor interactions and follow analogous tensor-transition rules controlled by Clebsch–Gordon-type rules. The underlying symmetry group SO_{2m} and the space $\mathbb{V}_{2m}$ in which it operates certainly have physical significance; we offered an interpretation.

References

[1] Byrd M S and Khaneja N 2003 Characterisation of the positivity of the density matrix in terms of the coherence representation *Phys. Rev.* A **68** 062322

[2] Kimura G 2003 The Bloch vector for *N*-level systems *Phys. Lett.* **314** 339

[3] Lawson H B and Michelsohn M-L 1989 *Spin Geometry* (Princeton, NJ: Princeton University Press) Wikipedia, Mathematics, Clifford Algebra

Lounesto Pertti 2001 *Clifford Algebras and Spinors* 2nd edn (Cambridge: Cambridge University Press)

[4] Dietz K 2006 Generalized Bloch spheres for *m*-qubit states *J. Phys. A: Math. Gen.* **39** 1433